

NEW DEVELOPMENTS OF THE ODDS THEOREM

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ABSTRACT. The odds theorem and the corresponding solution algorithm (odds algorithm) are tools to solve a wide range of optimal stopping problems. Its generality and tractability have caught much attention. (Google for instance “Bruss odds” to obtain a quick overview.) Many extensions and modifications of this result have appeared since publication in 2000. This article reviews the important new developments and applications in this field. The spectrum of application comprises as different fields as secretary problems, more general stopping problems, robotic maintenance problems, compassionate use clinical trials and others.

This review also includes a new contribution of our own.

1. THE ORIGINAL ODDS THEOREM

The odds-theorem is a result in the theory of optimal stopping which can be applied for many interesting sequential decision problems. The original version of the Odds-algorithm is the work of Bruss (2000). He discovered it when he saw common features between quite different looking optimal stopping problems.

The framework is as follows. There are n random variables which are observed sequentially. It is desired to stop online with maximum probability on a last specific event. No recall is permitted. Here “specific” is understood as being defined in terms of the interest of the decision maker. Such problems can be readily translated into the equivalent problem of stopping on a sequence of independent indicators. The parameters of those indicators variables are supposed to be known. Maximizing the resulting objective function means then maximizing the probability of selecting the last indicator of a specific event to be equal to 1 from the sequence. For convenience a variable equal to one will be called a *success*.

The independence property can be relaxed but is, at least locally, important. This is the main reason why the optimal strategy turns out to be a threshold rule based on a fixed time index.

We first recall the odds theorem, upon which the Odds-algorithm is based.

1991 *Mathematics Subject Classification.* 60G40.

Key words and phrases. Odds algorithm, records, secretary problems, robotic maintenance, clinical trials, investment problems, multiplicative odds, Markov chains.

Theorem 1 (Odds-Theorem, Bruss (2000)). *Let I_1, I_2, \dots, I_n be n independent Bernoulli random variables, with n known. We denote $(i = 1, 2, \dots, n)$ p_i the parameter of the variable I_i ($p_i \in [0, 1]$). Let $q_i = 1 - p_i$ and $r_i = p_i/q_i$. Define the index*

$$s = \begin{cases} \max \left\{ k \in \{1, 2, \dots, n\} : \sum_{j=k}^n r_j \geq 1 \right\} & \text{if } \sum_{i=1}^n r_i \geq 1, \\ 1 & \text{otherwise.} \end{cases}$$

To maximize the probability of stopping on the last “1” of the sequence it is optimal to stop on the first “1” that we meet among the variables I_s, I_{s+1}, \dots, I_n .

Based on the this theorem, we can describe the *odds-algorithm* as follows:

- 1° Compute the odds r_j for $j = n, n-1, \dots$ successively; compute the threshold s easily by looking at the running sum $r_n + r_{n-1} + \dots + r_j$ and stop as soon as this sum reaches or exceeds 1. This defines the stopping threshold s with $n-s+1$ further variables to come. We then must wait for the first $k \geq s$ (if any) with $I_k = 1$. Otherwise we must stop at index n anyway.
- 2° The optimal win probability is given by $V(n) = R_s Q_s$, where $Q_s = \prod_{j=s}^n q_j$ and $R_s = \sum_{j=s}^n r_j$.

A subsequent article of Bruss (2003) gives lower and upper bounds for the quantity $V(n)$.

The algorithm provides thus the optimal strategy τ_s and is optimal with respect to other considerations: linear complexity, ease of computation. Indeed with simple values of p_k this can even be computed by head.

Examples of applications. This result has been immediately noticed for its simplicity and generality. Several generalizations have appeared since then. The overview of these works is the content of the next sections. We recall first a few applications included in Bruss (2000).

1) *Classical Secretary Problem.* An interviewer has an appointment with each of the n (fixed) secretaries who are applying for a certain job. The secretaries' quality are independent of each other. He is not able to quantify the quality of a secretary but he is able to rank them relatively to each other. That is, after having observed, say, k secretaries one after the other he can compute their relative ranks, and in particular he is able to remember which one was the best candidate among these first k secretaries.

Solving this problem with the odds theorem is straightforward. An observation at time k is a record if and only if his relative rank among the first k variables is 1. It is known that $p_k = 1/k$. Hence $q_k = (k-1)/k$ and $r_k = 1/(1-k)$. The s index in the odds theorem can be computed and gives the optimal strategy. It is the largest s such that $\sum_{j=s}^n (j-1)^{-1} \geq 1$.

Therefore $V(n) = \frac{s-1}{n} \sum_{j=s-1}^{n-1} j^{-1}$. Note that $s = s(n) \sim n/e$ and thus $V(n) \rightarrow 1/e$. Indeed $V(n) \geq 1/e$ for all $n = 1, 2, \dots$.

We should also mention here that even if the parameter n is unknown, the best candidate can always be selected with probability of at least $1/e$, and this is a very tractable model. This is the so-called $1/e$ -law of Bruss (1984).

For a related problem see also Suchwałko and Szajowski (2003).

Remark. In the setting of the odds theorem, the real number $1/e$ is a lower bound for $V(n)$ provided that $\sum_{i=1}^n r_i \geq 1$ (see Bruss (2003)). This condition is always met in the classical secretary problem.

2) *Grouping observations.* Hsiao and Yang (2000) adress and solve secretary problem with group interviews. The selection is considered a success if the selected group that contains the best of all observations. This can also immediately be solved using the odds algorithm.

3) *Refusal probability.* It is easy to introduce an availability probability in the framework. If the probability for some variable I_k to be equal to 1 is p_k and, independently of the value of I_k , the variable is available with probability a_k , then the probability of having an available 1 is $\tilde{p}_k = p_k a_k$. The odds algorithm can compute the strategy based on the \tilde{p}_k 's.

4) *Unknown number of observations.* This model can immediately be generalized to a random number N of events as long as we assume independance of the indicators of the successes. For instance we model the unknown number N by a time-embedding through

$$p_k = P(I_k = 1 | \exists \text{ an observation at time } k) \cdot P(\exists \text{ an observation at time } k).$$

Dice game. A well-known game is the following. A die is thrown N times, N fixed. To one player it is asked to bet an amount of money one of the N throws. He wins if the die shows the value 6 at that time and if there is no more 6's in the following throws.

Since the probability of obtaining a 6 at any time is $1/6$, we input the values $(1/6, 1/6, 1/6, \dots)$ as the parameteres (p_k) in the algorithm. We obtain $s = N - 4$. That is, we will only look at the value shown by the last 5 dice and bet our money as soon as we see a 6.

Additional applications will be outlined in the last section.

2. STOPPING ON THE LAST SUCCESS: UNKOWN ODDS, RANDOM LENGTH, RECALL AND OTHER MODIFICATIONS

2.1. **Unkown odds.** This section addresses a difficult problem. What would we know if all we knew was that the observed variables are independent but have an unknown parameter? A natural approach would be to estimate the odds sequentially and to plug the estimates into the odds algorithm. Let us call such a strategy an “empirical odds strategy”.

For a detailed study of the performance of empirical odds strategies we refer to the paper of Bruss and Louchard (2009) in which they analyze and test several modifications.

The intuition tells us that the optimal strategy lies in the class of empirical strategies. But at the moment there is no theoretical result to show this. In a simple setup (small n) dynamic programming shows that this intuition is correct that is, that the empirical odds strategy is optimal. However for a larger n this is still an open problem, and an important one regarding applications, for it is often closer to reality than the model in which we assume that we know the parameters.

For motivation and examples of application for this model, see also Bruss (2006). In particular this also treats important applications in the domain of clinical trials.

2.2. Stopping on the m -th last success. The paper of Bruss and Paindaveine (2000) follows the spirit of the original odds-theorem of 2000. The setting is the same as in the previous section. The objective is now to predict the m -th last success upon its arrival, that is, to find the stopping time τ that maximizes the probability

$$(1) \quad P\left(\sum_{k=\tau}^n I_k = m\right)$$

and the optimal strategy associated with this win probability.

Theorem 2 (Bruss and Paindaveine (2000)). *An optimal rule for stopping on the m -th last success exists and is to stop on the first index (if any) with $I_k = 1$ and $k \geq s_m$ for some fixed $s_m \in \{1, 2, \dots, n - m + 1\}$.*

The s_m are computed as follows: define

$$\pi_k := \#\{j \geq k | r_j > 0\}$$

and

$$R_j^{(k)} = \sum_{k \leq i_1 < i_2 < \dots < i_j \leq n} r_{i_1} r_{i_2} \dots r_{i_j}.$$

We then have

$$(2) \quad s_m = \sup \left\{ 1, \sup \left\{ 1 \leq k \leq n - m + 1 : R_m^{(k)} \geq m R_{m-1}^k \text{ and } \pi_k \geq m \right\} \right\}.$$

It is to mention that unimodality of the optimal strategy is not straightforward and that it needs a delicate treatment. The stopping index s_m can be computed but has a slightly more sophisticated form than before.

2.3. Hsiao and Yang's Markovian framework.

Homogeneous case. Hsiao and Yang (2002) study a modification of the same model where now the I_1, I_2, \dots, I_N form a Markov chain. The authors prefer to renumerate the indicators backwards. Hence let $I_N, I_{N-1}, \dots, I_1, I_0$ be a Markov chain with the following structure:

$$\begin{aligned} P(I_{n-1} = 1 | I_n = 0) &= \alpha_n \\ P(I_{n-1} = 0 | I_n = 1) &= \beta_n \end{aligned}$$

The authors treat again the objective to stop with maximum probability on the last success. Let

$$S_n = P(I_j = 0, \forall j = n-1, \dots, 0 | I_n = 1) = \beta_n \prod_{i=1}^{n-1} (1 - \alpha_i)$$

and let

$$\begin{aligned} q_0(n) &= \text{optimal success probability on } I_{n-1}, \dots, I_0 \text{ given that } I_n = 0, \\ q_1(n) &= \text{optimal success probability on } I_{n-1}, \dots, I_0 \text{ given that } I_n = 1. \end{aligned}$$

It can be seen that the stopping time is defined as the first n such that

$$(3) \quad S_n \geq q_1(n),$$

and as 0 if there is no such n . Let $(\phi_j, j = N, \dots, 1, 0)$ be the stopping strategy. So this sequence is adapted to the process $(I_j, j = N, \dots, 1, 0)$; $\phi_j = 1$ means that we choose to stop at time j if $I_j = 1$ and $\phi_j = 0$ means that we continue observing more variables, whatever the value of I_j . We always have $\phi_0 = 1$ because in our problems a decision must be made within the set $\{N, N-1, \dots, 0\}$.

The first result is obtained in the case where α_n and β_n are constants, for all n ; we set $\alpha := \alpha_0$ and $\beta := \beta_0$.

Theorem 3 (Hsiao and Yang (2002)). *If $\beta \in [\frac{1}{2}, 1]$, then*

- (i) *if $\alpha = 0$, $\phi_j = 1$ for all $j = N, N-1, \dots, 0$;*
- (ii) *if $\alpha = 1$, $\phi_0 = \phi_1 = 1$ and $\phi_j = 0$ for $j = N, N-1, \dots, 2$;*
- (iii) *if $\alpha \in (0, 1)$, $\phi_j = 0$ for $j \in \{N, \dots, r+2, r+1\}$ and $\phi_j = 1$ for $j \in \{r, r-1, \dots, 0\}$, where $r = \min \{ \lfloor (\beta - 2\alpha)(1 - \alpha)/\alpha\beta \rfloor + 2, N \}$*

Therefore, there exists an r such that $\tau_N = \sup \{0 \leq i \leq N | I_i = 1, i \leq r\}$ with the convention that $\sup \emptyset = 0$.

The case which involves more calculations is the third. Hsiao and Yang obtain the explicit form of $q_1(k)$ and $q_0(k)$ by solving the recurrence defining those two functions. The index r is then obtained using (3) and replacing q_1 by its explicit value.

We should explain why this result holds for $\beta \in [\frac{1}{2}, 1]$. A high value for β means that once we observe the variable with value 1, it is likely that there will be other variables in the future equal to 0. On the other hand, a small β means that it is likely that after a 1, we have many variables being equal to 1.

Similar but more delicate cases arise for the case $\beta \in (0, \frac{1}{2})$. The resulting strategy in the non-degenerate case deserves also an explanation. Depending on whether the quantity

$$(4) \quad (\alpha + \beta)\beta(1 - \alpha)^n - \beta(1 - \alpha - \beta)^{n+1}$$

is smaller than α for any $n = 0, 1, \dots, N - 1$ or exceeds α for some n , the resulting strategy has a very different structure. We now state their result.

Theorem 4 (Hsiao and Yang (2002)). *Let $\beta \in (0, \frac{1}{2})$.*

- (i) *If the quantity defined in (4) is always smaller than α , we have $\phi_0 = 1$, $\phi_j = 0$ for all $j > 0$;*
- (ii) *If $\alpha \neq 0$ and there exists an $r \in \{N - 1, \dots, 1\}$ such that*

$$(5) \quad (\alpha + \beta)\beta(1 - \alpha)^r - \beta(1 - \alpha - \beta)^{r+1} \geq \alpha > (\alpha + \beta)\beta(1 - \alpha)^k - \beta(1 - \alpha - \beta)^{k+1}$$
for all $k < r$. Define

$$m = \left\lfloor \frac{(\alpha + \beta)(\alpha^2 - \alpha + \beta)(1 - \alpha)^{r-1} - \alpha[1 - (1 - \alpha - \beta)^{r+1}]}{\alpha\beta(\alpha + \beta)(1 - \alpha)^{r-1}} \right\rfloor + 1,$$

we have the following optimal strategy

$$\phi_j = \begin{cases} 1 & \text{for } j \in \{0, r + 1, r + 2, \dots, r + m\} \\ 0 & \text{else} \end{cases} \quad \text{if } r + m < N,$$

$$\phi_j = \begin{cases} 1 & \text{for } j \in \{0, r + 1, r + 2, \dots, N\} \\ 0 & \text{else} \end{cases} \quad \text{if } r + m \geq N.$$

This is a case where the optimal stopping has more than one stopping island.

- (iii) *If (5) is verified for some r for all $k < r$ and if $\alpha = 0$, then there exists $r < N$ such that*

$$\phi_j = \begin{cases} 1 & \text{for } j \in \{0, r, r + 1, r + 2, \dots, N\} \\ 0 & \text{for } j \in \{1, 2, \dots, r - 1\} \end{cases}.$$

This strategy represents the stopping time τ_N defined as

$$\tau_N = \sup \{0 \leq i \leq N \mid I_i = 1, i = 0 \text{ or } i > r\}.$$

The probability of selecting the last success by using the optimal strategy can be computed for all β .

Nonhomogeneous case. Hsiao and Yang then study the corresponding non-homogeneous case and obtain the following theorem under some assumptions.

Theorem 5 (Hsiao and Yang (2002)). *If $\alpha_n + \beta_n \geq 1$ for all n then*

$$\phi_j = \begin{cases} 1 & \text{for } j \in \{0, 1, \dots, r\} \\ 0 & \text{for } j \in \{r + 1, r + 2, \dots, N\} \end{cases}$$

where

$$r = \inf \left\{ k \in \mathbb{N}_N \cup \{0\} : \sum_{l=1}^k \frac{\alpha_l \beta_{l-1}}{(1-\alpha_l)(1-\alpha_{l-1})} + \frac{\beta_k(1-\beta_{k+1})}{\beta_{k+1}(1-\alpha_k)} > 1 \right\}.$$

We can remark that in their results the optimal strategy cannot be simplified into a “sum-the-odds” strategy (any kind of odds).

The optimal strategy may have now a different form. One can see that, as pointed out in the theorem, there can be more than just one stopping island.

2.4. Tamaki’s Markovian result. Tamaki (2006) tackles a similar problem as in the previously exposed Markovian framework of Hsiao and Yang. There are important differences, however. First the hypotheses on the transition probabilities are different. Second, his objective is to obtain a solution relying on a sum of odds.

Let I_1, I_2, \dots, I_n be a sequence of independent indicator variables. Let us study the following Markov dependence between the variables:

$$\alpha_j = P(I_{j+1} = 1 | I_j = 0)$$

$$\beta_j = P(I_{j+1} = 0 | I_j = 1)$$

for $1 \leq j \leq n-1$. And let us assume that $\alpha_n = 0$, $\beta_n = 1$. We write $\bar{\alpha}_j$ and $\bar{\beta}_j$ for $1 - \alpha_j$ and $1 - \beta_j$ respectively. The result is as follows:

Theorem 6 (Tamaki (2006)). *Assume that*

- a) (α_j) is non-increasing in j ,
- b) (β_j) is non-decreasing concave in j .

Then an optimal rule stops on the first index $k \geq s$ such that $I_k = 1$ and where

$$s = \sup \left\{ 1 \leq k \leq n : \frac{\bar{\beta}_k \beta_{k+1}}{\beta_k \bar{\alpha}_{k+1}} + \sum_{j=k+1}^{n-1} \frac{\alpha_j \beta_{j+1}}{\bar{\alpha}_j \bar{\alpha}_{j+1}} \geq 1 \right\}$$

with the natural convention that the empty sum equals 0.

2.5. Multiple sum-the-odds theorem (Ano et al., 2010). Suppose that we are given $m \in \mathbb{N}$ selection chances in the problem described in the preceding section. Let $V_i^{(m)}$, $i \in \mathbb{N}$, denote the conditional maximum probability of win provided that we observe $X_i = 1$ and select this success when we have at most m selection chances left. Let $W_i^{(m)}$, $i \in \mathbb{N}$, denote the conditional maximum probability of win provided that we observe $X_i = 1$ and ignore this success when we have at most m selection chances left. Let, furthermore, $M_i^{(m)}$, $i \in \mathbb{N}$, denote the conditional maximum probability of win provided that we observe $X_i = 1$ and are faced with a decision to select or not when we have at most m selection chances left. The optimality equation is then given by

$$M_i^{(m)} = \max \left\{ V_i^{(m)}, W_i^{(m)} \right\}, \quad i \in \mathbb{N}.$$

For each $i \in \mathbb{N}$, define recursively the quantities $H_i^{(m)}$ by

$$(6) \quad H_i^{(1)} = 1 - \sum_{j=i+1}^N r_j,$$

$$(7) \quad H_i^{(m)} = H_i^{(1)} + \sum_{j=(i+1) \vee i_*^{(m-1)}}^N r_j H_j^{(m-1)},$$

where $i_*^{(m)} = \min \{i \in \mathbb{N} : H_i^{(m)} > 0\}$.

Now the theorem in Ano et al. reads:

Theorem 7 (Ano, Kakinuma, Miyoshi (2010)). *Suppose that we have at most $m \in \mathbb{N}$ selection chances. Then, the optimal selection rule $\tau_*^{(m)}$ is given by*

$$(8) \quad \tau_*^{(m)} = \min \{i \geq i_*^{(m)} : X_i = 1\}$$

where $\min \emptyset = +\infty$. Furthermore, we have

$$(9) \quad 1 \leq i_*^{(m)} \leq i_*^{(m-1)} \leq \dots \leq i_{*(1)} \leq N.$$

It would be interesting to have an intuitive understanding of the quantities $H_i^{(m)}$, but this seems difficult.

In Ano and Matsui (2012), a lower bounds for the multiple stopping problem is obtained.

2.6. Random Length. Tamaki, Wang and Kurushima (2008) allow random length and provide a sufficient condition for the optimal rule to be of threshold type.

Ano, in a preprint (2011), tackles again the multiple stopping problem, with random length.

Random length and refusal probability at the same time are studied in Horiguchi and Yasuda (2009).

2.7. Ferguson's modification of the Odds-Theorem. Ferguson (2008) proposed the following modification of the original odds-theorem described in section 1.

Let Z_1, Z_2, \dots be a stochastic process on an arbitrary space with an absorbing state called 0. For $i = 1, 2, \dots$, let Z_i denote the set of random variables observed after success $i - 1$ up to and including success i . If there are less than i successes, we let $Z_i = 0$, where 0 is a special absorbing state. The general model is as follows.

We make the assumption that with probability one the process will eventually be absorbed at 0. We observe the process sequentially and wish to predict one stage in advance when the state 0 will first be hit. If we predict correctly, we win 1, if we predict incorrectly we win nothing, and if the

process hits 0 before we predict, we win ω , where $\omega < 1$. This is a stopping rule problem in which stopping at stage n yields the payoff

$$(10) \quad \begin{aligned} Y_n &= \omega I(Z_n = 0) + I(Z_n \neq 0)P(Z_{n+1} = 0|\mathcal{G}_n) \quad \text{for } n = 1, 2, \dots \\ Y_\infty &= \omega \end{aligned}$$

where $\mathcal{G}_n = \sigma(Z_1, \dots, Z_n)$, the σ -field generated by Z_1, \dots, Z_n . The assignment $Y_\infty = \omega$ means that if we never stop, we win ω .

The resulting *one-stage look-ahead rule* (1-sla) is to stop at index N defined by

$$(11) \quad N := \min \{k : Z_k = 0 \text{ or } (Z_k \neq 0 \text{ and } W_k/V_k \leq 1 - \omega)\}$$

where

$$\begin{aligned} V_k &= P\{Z_{k+1} = 0|\mathcal{G}_k\}, \\ W_k &= P\{Z_{k+1} \neq 0, Z_{k+2} = 0|\mathcal{G}_k\}. \end{aligned}$$

The event $\{Z_{k+1} \neq 0, Z_{k+2} = 0\}$ given the history \mathcal{G}_k describes the event that there is exactly one success in the future because 0 is an absorbing state. A sufficient condition for the problem to be monotone is, as Ferguson shows,

$$(12) \quad W_k/V_k \text{ is a.s. non-increasing in } k.$$

Theorem 8 (Ferguson (2008)). *Suppose that the process Z_1, Z_2, \dots has an absorbing state 0 such that $P(Z_k \text{ is absorbed at } 0) = 1$ and that the stopping problem with reward sequence (10) satisfies the condition (12). Then the 1-sla is optimal.*

This model enables us to tackle more general problems. The level of generality and abstraction of this model makes it a very tractable result. Furthermore, Ferguson's paper contains several examples for which the 1-sla rule turns out to be a sum-the-odds strategy.

3. THE ROLE OF THE k -FOLD MULTIPLICATIVE ODDS

As before in the paper of Bruss and Paindaveine (2000) where the authors considered a group of last successes, we have here again to deal with multiplicative odds. We now present the problem studied by Tamaki (2000).

The problem is the following. Find the strategy that maximizes the probability of stopping on any of the last m successes. All hypotheses are the same as the ones mentioned in the original odds theorem in Section 1.

In Ferguson's framework from Section 2.7, the current problem leads us to consider the following payoffs

$$Y_k = I(Z_k \neq 0)P(Z_{k+m} = 0|\mathcal{F}_k), \quad k = 1, 2, \dots, n,$$

where \mathcal{F}_k is the σ -field generated by Z_1, Z_2, \dots, Z_k , and the following quantities

$$\begin{aligned} V_k &= P(Z_{k+1} = 0 | \mathcal{F}_k), \\ W_k &= P(Z_{k+m} \neq 0, Z_{k+m+1} = 0 | \mathcal{F}_k). \end{aligned}$$

A corollary of Theorem 8 from Section 2.7 is as follows:

Corollary 1 (Ferguson (2008)). *Suppose that n Bernoulli random variables X_1, X_2, \dots, X_n are observed sequentially. Let $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n$ be an increasing sequence of sigma-fields such that $\{X_j = 1\} \in \mathcal{F}_j$ for all $1 \leq j \leq n$. Let*

$$\begin{aligned} V_k &= P(X_{k+1} + \dots + X_n = 0 | \mathcal{F}_k), \\ W_k &= P(X_{k+1} + \dots + X_n = m | \mathcal{F}_k). \end{aligned}$$

Then the optimal rule is determined by the stopping time

$$N_m = \min \{k \geq 1 : X_k = 1 \text{ and } W_k/V_k \leq 1\},$$

provided that the sequence $(W_k/V_k, k \geq 1)$ is monotone non-increasing.

Set, as above,

$$(13) \quad R_j^{(k)} = \sum_{k \leq i_1 < i_2 < \dots < i_j \leq n} r_{i_1} r_{i_2} \dots r_{i_j}.$$

The result is a sum-the-odds strategy, but the odds are the multiplicative odds given in (13).

Theorem 9 (Tamaki (2010)). *For the stopping problem of maximizing the probability of stopping on any of the last m successes in n independent Bernoulli trials, the optimal rule stops on the first success $X_k = 1$ with $k \geq s_m$, if any, where*

$$s_m = \min \left\{ k \geq 1 : R_m^{(k+1)} \leq 1 \right\}.$$

Moreover, the maximal probability of win is

$$v_m = \left(\prod_{j=s_m}^n q_j \right) \left(\sum_{j=s_m}^n R_j^{s_m} \right).$$

Remark. The two problems we mentioned here involve the m last 1's of the sequence. It would be interesting to know in advance if a particular problem will be a sum-the-odds theorem involving the multiplicative odds.

We now include a new contribution.

4. A NON-INFORMATIVE PROBLEM IN CONTINUOUS TIME

It is possible to translate the problem described in section 2.1 to a continuous setting as follows. Let $I_1, I_2, I_3 \dots$ be independent indicator variables with a common parameter $p \in (0, 1]$. But here we suppose to have absolutely no information about the parameter p .

Consider an homogeneous Poisson process with rate 1 on $[0, T]$. Let N_t the number of indicator variables observed up to time t . So N_t counts that number of points in the Poisson process in which an indicator is observed. We want to stop on the last indicator equal to 1 which arrived in the time interval $[0, T]$.

It is well-known that independent thinning of Poisson process is again a Poisson process. Hence the arrival process of the 1's is a Poisson process with unknown rate p . Let \tilde{N} denote the thinned Poisson process of successes. Here \tilde{N}_t counts the number of points in the Poisson process in which an indicator variable of value 1 is observed. We thus want to stop on the last arrival of this process \tilde{N} in the interval $[0, T]$. Here we suppose that only this process (\tilde{N}_t) is observable.

We follow the approach of the recent paper of Bruss and Yor (2012) to the so-called *last arrival problem* and derive the optimal strategy.

Noting that $E[\tilde{N}_t] = pE[N_t] = pt$ we can follow the reasoning of Bruss and Yor to conclude that, for all $s, t > 0$, the process (\tilde{N}_t) must satisfy

$$(14) \quad E[\tilde{N}_{t+s} - \tilde{N}_t | \mathcal{F}_t] = sp = \frac{s}{t}pt,$$

where \mathcal{F}_t denotes the filtration defined by $\mathcal{F}_t = \sigma\{\tilde{N}_u : 0 < u \leq t\}$. Bruss and Yor (2012) called such a process a p.i.-process, that is a process with proportional increments. Such a process must satisfy

$$(15) \quad E(\tilde{N}_{t+s} - \tilde{N}_t | \mathcal{F}_t) = \frac{s}{t}\tilde{N}_t \text{ a.s.}$$

According to Theorem 1 from Bruss and Yor (see page 3243), (\tilde{N}_t/t) is a \mathcal{F}_t -martingale for $t \geq \tilde{T}_1$, where \tilde{T}_1 is the first jump of \tilde{N}_t . (Note that in the special case where if $\tilde{T}_1 > T$, there is no "1" in the interval $[0, T]$ and we lose by definition.)

Furthermore we note that (\tilde{N}_t) and (\tilde{N}_t/t) have exactly the same jump times. Since $E(\tilde{N}_t) = pt$ we have $E(\tilde{N}_t/t) = p$, and from the martingale property $E(\tilde{N}_T/T) = E(\tilde{N}_t/t) = p$. Exactly as in Bruss and Yor (2012) it is then optimal to stop at the k -th arrival time if and only if

$$(16) \quad k \left(\frac{T - \tilde{T}_k}{\tilde{T}_k} \right) \leq 1.$$

And if there is no arrival time \tilde{T}_k in $[0, T]$ for which this condition is verified such k then we lose. \square

Remark. This is, as far as we are aware, the only case when the unknown odds-problem allow for a solution which is proved to be optimal. It would be interesting to know whether the criterion (16) would also be optimal if both (N_t) and (\tilde{N}_t) are observable, because in this case the relevant filtration would be the larger filtration generated by $(N_u)_{u \leq t}$ and the indicators seen up to time t .

5. APPLICATIONS

We now give other important applications which can be solved by the odds algorithm or its newer developments.

5.1. The ballot problem. Tamaki (2001) considers the problem of stopping on the maximum point of a random trajectory. One of the models he presents is quickly solved by the odds theorem. The paper studies also several other problems and contains interesting ideas for future research.

5.2. Online Calibration in Local Search (Bontempi, 2011). To search the minimum of a real-valued function a computer creates a grid of points in the domain of the function and evaluates the function in each of these points. If the function does not fluctuate too much we will look at the point giving the smallest value for our function and think that this point should be close to the real minimum of the function into the considered domain.

When the grid is really tight we evaluate many points and our estimate for the minimum becomes better. But this is computationnally unefficient, and one prefers stochastic approaches.

Bontempi suggests to start from an initial best point x_0 and try random points in the neighbourhood of x_0 . If the function f evaluates smaller in one of these points, say x_1 , *follow the path* $x_0 \rightarrow x_1$ and hope that there is another better point in the neighbourhood of x_1 . This is the new current best point (or solution).

When to stop? It would be best if one could stop searching when being on the very best solution. If the neighbourhood of some x_k did not give better points, go back to x_{k-1} and investigate the second best solution's neighbourhood. This might give a better solution but also might not. If this *go backward-take next best* procedure does not give a better solution, then x_k was indeed the best solution in the neighbourhood of x_{k-1} but also the best of all x_1, x_2, \dots, x_k . So this a good candidate for a minimum of f .

5.3. Automation and Maintenance (Iung, Levrat, Monnin, Thomas (2006, 2007, 2008)). The research group of Iung, Monnin, Levrat and Thomas of the *Centre de Recherche en Automatique de Nancy*, CNRS, France, has applied the odds algorithm to problems of *automation and maintenance*. This application is intended to provide a strategy for a maintenance tool (a robot) to choose which part of a system to replace if there is more than one failure. The choice would be based on the life expectation of this

piece of the system, the time used to replace it and the probability of breakdown of it.

Putting the maintenance problem into the *odds framework* is a most interesting task (see [27], [28], [29] and [30]) with many challenging questions.

The Nancy research group used the odds algorithm to formulate a strategy to select the priorities of replacements of parts. They were aware of the fact that the independence condition needed in the hypotheses is not always satisfied, because failures may be dependent of each other.

5.4. Odds and software. In Skroch and Turowski (2010), the odds algorithm is used as a decision tool for optimal selection when a maintenance task must be performed on particular software systems.

The authors explain that advanced software systems can reconfigure themselves at run-time by choosing between alternative options for performing certain functions and that such options can be built into the systems. However, they point out that these software systems are also externally available on open and uncontrolled platforms, such as Web services and mashups on the Internet.

The authors show how run-time software self-adaptation with uncontrolled external options can be optimized by stopping theory, yielding the best possible lower probability bound for choosing an optimal option.

5.5. Investment models (Bruss and Ferguson (2002)). In a venture capital investment, one wants to invest a certain amount of money in, for example, a particular domain of technology. This money often must have been placed before a fixed date, and therefore it is highly preferable that the best innovation within this period is the one that was chosen.

It is not too hard to see that again we are here waiting for some particular event among all observable events, where we try to detect the last one, when it happens. The event of interest would here be described as follows: today is an “opportunity” if today’s technology is better than the ones we observe since the beginning of our observing period. Call this “opportunity” a success and a day without opportunity a “failure”, or respectively write 1 and 0. This is almost an “odds-theorem” setting.

For other high risk investment models, see also Lebek and Szajowski (2007).

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